A potential lagrangian formulation of ideal MHD

Bruno Després

A potential lagrangian formulation of ideal MHD

Bruno Després

January 12, 2010

Résumé On propose une reformulation du système de la magnétohydrodynamique idéale en variable de Lagrange comme un système étendu de type hyperélastique. L'hyperbolicité est étudiée au travers du tenseur acoustique qui est strictement positif pour toute direction non orthogonale au champ magnétique. L'extension à la formulation eulérienne est présentée ainsi que quelques pistes pour la définition de méthodes numériques.

Abstract

We propose a reformulation of ideal magnetohydrodynamics written in the lagrangian variable as an enlarged system of hyperelastic type, with a specific potential. We study the hyperbolicity of the model and prove that the acoustic tensor is positive for all directions which are non orthogonal to the magnetic field. The consequences for eulerian ideal magnetohydrodynamics and for numerical discretization are briefly discussed at the end of this work.

1 Introduction

Ideal magnetohydrodynamics models the strong interaction of a charged fluid with an electromagnetic field, see [16, 15]. It has fundamental applications in computational plasma physics and for astrophysical simulations. The model admits a fully conservative formulation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} + p\mathbf{I} + \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0} \right) = 0, \mu = 4\pi, \\ \partial_t(\rho e + \frac{|\mathbb{B}|^2}{2\mu_0}) + \nabla \cdot \left(\rho \mathbf{u} e + p\mathbf{u} + \frac{(\mathbf{u}\wedge\mathbf{B})\wedge\mathbf{B}}{\mu_0} \right) = 0, \\ \partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = 0. \end{cases} \tag{1}$$

This system is supplemented by the divergence free condition for the magnetic field

$$\nabla \cdot \mathbf{B} = 0. \tag{2}$$

This constraint is actually an involutive equation, that is it is automatically satisfied for all $t > 0$ provided it holds at $t = 0$. On a physical ground this relation is a fundamental one and is part of the system. The free divergence constraint is closely connected to the entropy balance. For smooth solutions one can prove that

$$\partial_t(\rho S) + \nabla \cdot (\rho S \mathbf{u}) = -\frac{(\mathbf{B}, \mathbf{u})}{\rho T} \nabla \cdot \mathbf{B}. \tag{3}$$

Here S is the physical entropy. It is a function of the internal energy $\varepsilon = e - \frac{1}{2}|\mathbf{u}|^2$ and of the specific volume $\tau = \frac{1}{\rho}$, such that the second principle of the thermodynamics $TdS = d\varepsilon + pd\tau$ holds. Of course the right hand side in (3) vanishes for magnetic fields (2).

A considerable attention has been paid recently to the analysis of (1) since the understanding of the structure of this system is the key for a coherent, stable and consistent discretization. We refer to [18, 20, 19, 6]. See also [10, 2] for a discretization of lagrangian ideal MHD. A comprehensive reference is the chapter about ideal MHD in the monograph [15] and references therein. In most of these references one uses the eigenstructure of (1) to construct Riemann solvers. The multi-dimensional case reveals to be much more difficult, essentially because one cannot eliminate (2) by simple means as in dimension one. It has the consequence that the eigenstructure of (1) is spoiled, and missing eigenvectors make the design of Riemann solver problematic. The seminal paper of Powell [19] is a good example. In this reference the author modifies (1) with a formally non zero right hand side

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} + p\mathbf{I} + \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0} \right) = -\mathbf{B} \nabla \cdot \mathbf{B}, \\ \partial_t(\rho e + \frac{|\mathbb{B}|^2}{2\mu_0}) + \nabla \cdot \left(\rho \mathbf{u} e + p\mathbf{u} + \frac{(\mathbf{u}\wedge\mathbf{B})\wedge\mathbf{B}}{\mu_0} \right) = (\mathbf{u}, \mathbf{B}) \nabla \cdot \mathbf{B}, \\ \partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = -\mathbf{u} \nabla \cdot \mathbf{B}. \end{cases} \tag{4}$$

The entropy law becomes

$$\partial_t(\rho S) + \nabla \cdot (\rho S \mathbf{u}) = 0 \tag{5}$$

for smooth solutions. This eigenstructure of this system now admits a full set of real eigenvalues and eigenvectors, but at the price of adding a formally non zero right hand side. By comparisons of (1-3) and (4-5) it seems that one has to choose between a conservative formulation (1) with a non conservative entropy condition (3) and a non conservative formulation (4) endowed with a conservative entropy condition (5). At the continuous level it makes no harm since the involutive condition holds true. The real problem is a the discrete level since the discrete preservation of the free divergence constraint is difficult to enforce. This issue is also linked to what it called divergence cleaning, which is a numerical procedure to insure the discrete satisfaction of the free divergence condition. Some references are in [8, 23, 4, 15, 3]. To our knowledge this problem has not been solved in a definitive manner, despite constant efforts. See also [5, 21, 13] where the the compatibility of linear and non-linear Maxwell's equations is studied in conjunction with the free divergence problem.

In this work we study a reformulation of (1) in Lagrange coordinates by rewriting ideal MHD as an hyperelastic like model (8) with a new potential.

This new formulation sheds new light on a principle that was proposed by Godunov [11, 12] about the advantage of having a common formulation of both ideal MHD and elasticity. In the classical presentation [15], ideal MHD and hyperelastic models are close but they are fundamentally different. With the formulation proposed in this work, they share the same potential formulation: only the potential changes. The price is to enlarge the size of the system (1) by incorporating the gradient of deformation in the unknowns. The main result of this work will be a proof that the satisfaction of the entropy criterion is true for the conservative lagrangian formulation without any condition on the free divergence of the magnetic field. This is stated in theorem 6. It shows a fundamental difference between lagrangian formulations of ideal MHD and eulerian formulations.

The notations used in this work are standard. However we recall them to have a coherent set of notations for tensors. The gradient of a vectorial function $\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$ is $\nabla \mathbf{E} = \begin{pmatrix} \partial_{x_1} E_1 & \partial_{x_2} E_1 & \partial_{x_3} E_1 \\ \partial_{x_1} E_2 & \partial_{x_2} E_2 & \partial_{x_3} E_2 \\ \partial_{x_1} E_3 & \partial_{x_2} E_3 & \partial_{x_3} E_3 \end{pmatrix}$. The curl of $\mathbf{E}$ is $\nabla \wedge \mathbf{E} = \begin{pmatrix} \partial_{x_2} E_3 - \partial_{x_3} E_2 \\ -\partial_{x_3} E_1 + \partial_{x_1} E_3 \\ \partial_{x_1} E_2 - \partial_{x_2} E_1 \end{pmatrix}$. The divergence of $\mathbf{E}$ is noted $\nabla \cdot \mathbf{E}^t = \partial_{x_1} E_1 + \partial_{x_2} E_2 + \partial_{x_3} E_3 = \mathrm{tr}\,(\nabla \mathbf{E})$. The notation $\nabla \cdot \mathbf{E}^t$ is for the sake of compatibility with the divergence of a matrix that will be introduced below. So the free divergence condition for $\mathbf{B}$ will now be written $\nabla \cdot \mathbf{B}^t = 0$. The contraction of two tensors (matrices), $\mathbf{M} = (m_{ij})$ and $\mathbf{N} = (n_{ij})$, is $\mathbf{M} : \mathbf{N} = \sum_{i,j} m_{ij} n_{ij}$. The tensor product of vectors $\mathbf{D}$ and $\mathbf{E}$ is $\mathbf{D} \otimes \mathbf{E} = \begin{pmatrix} D_1 E_1 & D_2 E_1 & D_3 E_1 \\ D_1 E_2 & D_2 E_2 & D_3 E_2 \\ D_1 E_3 & D_2 E_3 & D_3 E_3 \end{pmatrix}$. The product of vectors is $\mathbf{D} \wedge \mathbf{E} = \begin{pmatrix} D_2 E_3 - D_3 E_2 \\ D_3 E_1 - D_1 E_3 \\ D_1 E_2 - D_2 E_1 \end{pmatrix}$. As a consequence $\nabla \wedge (\mathbf{D} \wedge \mathbf{E}) = \nabla \cdot (\mathbf{E} \otimes \mathbf{D} - \mathbf{D} \otimes \mathbf{E})$. Since $\nabla \cdot (\mathbf{D} \otimes \mathbf{E}) = (\nabla \cdot \mathbf{D}^t)\,\mathbf{E} + (\nabla \mathbf{E})\,\mathbf{D}$, the magnetic equation can be rewritten as

$$\partial_t \mathbf{B} + (\nabla \mathbf{B})\,\mathbf{u} = \left(\nabla \cdot \mathbf{B}^t\right)\mathbf{u} + (\nabla \mathbf{u})\,\mathbf{B} - \left(\nabla \cdot \mathbf{u}^t\right)\mathbf{B}$$

from which we deduce the classical relation

$$D_t \mathbf{B} = (\nabla \mathbf{u})\,\mathbf{B} - \left(\nabla \cdot \mathbf{u}^t\right)\mathbf{B}, \qquad D_t = \partial_t + \mathbf{u} \cdot \nabla, \tag{6}$$

since $\mathbf{B}$ is divergence free. The physical interpretation is that the material derivative of the magnetic field does not depend on the spatial derivative of the magnetic field. The notation $\mathbf{M}^{-t}$ stands for the inverse of the transpose of $\mathbf{M}$, that is $\mathbf{M}^{-t} = \left(\mathbf{M}^{-1}\right)^t$.

The plan of this work is as follows. We will begin by rewriting ideal MHD like an hyperelastic model (8) with a new potential. The hyperbolicity will be established starting from (8). Then we will establish a non-conventional eulerian formulation of ideal MHD. We will conclude by some numerical perspectives.

2 Lagrangian potential ideal MHD

We will make a parallel between the structure of ideal MHD and the structure of hyperelastic models. Using the Lagrange variable $\mathbf{X}$ which is defined by the Lagrange-Euler transformation

$$\mathbf{x}'(t) = \mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{X}, \tag{7}$$

an hyperelastic model for compressible isotropic hyperelasticity writes

$$\begin{cases} D_t\rho_0 = 0, \\ D_t\mathbf{F} = \nabla_{\mathbf{X}}\mathbf{u}, \\ D_t\left(\rho_0\mathbf{u}\right) = \nabla_{\mathbf{X}} \cdot \left(\rho_0 \nabla_{\mathbf{F}}\varphi\right), \\ D_t\left(\rho_0 e\right) = \nabla_{\mathbf{X}} \cdot \left(\rho_0 \mathbf{u}^t \nabla_{\mathbf{F}}\varphi\right). \end{cases} \tag{8}$$

The initial density is ρ_0. The notation D_t represents the material derivative, which is also the partial derivative with time is we consider that $\mathbf{X}$ is frozen. That is $D_t = \partial_t + \mathbf{u}\cdot\nabla = \partial_{t|\mathbf{X}}$. The function $\mathbf{F} \mapsto \varphi(\mathbf{F}, S; \rho_0)$ is a potential and S is the entropy of the system. Another equation of the hyperelastic model is $\varphi(\mathbf{F}, S, \rho_0) = e - \frac{1}{2}\left|\mathbf{u}\right|^2$ which means that $\varphi(\mathbf{F}, S)$ is the internal energy. The theory of hyperelastic models is developed for example in the recent work [22] and references therein. See also the monograph [7]. Based an invariance considerations, (8) must compatible with the group of rotations which means that

$$\varphi(\mathbf{F}, S, \rho_0) = \psi(i_1, i_2, i_3, S, \rho_0) \tag{9}$$

where i_p is the invariant of degree p of the Finger Cauchy-Green tensor $\mathbf{C}$, that is

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3,\ i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_2,\ i_3 = \lambda_1\lambda_2\lambda_3$$

where the λ_j are the eigenvalues of $\mathbf{C} = \mathbf{F}^t\mathbf{F}$. The force that acts on the right hand side of the impulse equation is the first Piola-Kirkhhoff tensor

$$\sigma^{CG} = \rho_0 \nabla_{\mathbf{F}}\varphi. \tag{10}$$

We now use this formalism and first rewrite (1) using the Lagrange variable $\mathbf{X}$.

Proposition 1. *For all vector fields $\mathbf{G}$, one has the formula*

$$J\nabla_{\mathbf{x}} \cdot \mathbf{G}^t = \nabla_{\mathbf{X}} \cdot \left(\mathbf{G}^t \mathrm{cof}\left(\mathbf{F}\right)\right) \tag{11}$$

where $J = det(\mathbf{F})$ and $cof(\mathbf{F})$ is the comatrix, that is the matrix of the cofactors.

This Piola formula is true in the weak sense [7] in all dimension. $\square$

It is immediate to show that the impulse and energy equations in (1) can be rewritten as

$$\begin{cases} \rho D_t\mathbf{u} = -\nabla_{\mathbf{x}} \cdot \left(p\mathbf{I} + \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0}\right), \\ \rho D_t\left(e + \frac{|\mathbb{B}|^2}{2\mu_0\rho}\right) = -\nabla_{\mathbf{x}} \cdot \left(\mathbf{u}^t\left(p\mathbf{I} + \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0}\right)\right). \end{cases} \tag{12}$$

Multiplying by $J = \det(\mathbf{F}) = \frac{\rho_0}{\rho}$, one gets the equations in the Lagrange frame

$$\begin{cases} D_t(\rho_0 \mathbf{u}) = -\nabla_{\mathbf{X}} \cdot \left(\left(p\mathbf{I} + \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0} \right) \operatorname{cof}(\mathbf{F}) \right), \\ D_t \left(\rho_0 e + \rho_0 \frac{|\mathbb{B}|^2}{2\mu_0 \rho} \right) = -\nabla_{\mathbf{X}} \cdot \left(\mathbf{u}^t \left(p\mathbf{I} + \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0} \right) \operatorname{cof}(\mathbf{F}) \right). \end{cases} \tag{13}$$

In principle we could apply the same method for the magnetic field $\mathbf{B}$ in (1). We will not do that. Instead we remark that $\mathbf{B}$ is divergence free in the $\mathbf{x}$ frame. The application of formula (11) to $\nabla_{\mathbf{x}} \cdot \mathbf{B}^t = 0$ yields

$$\nabla_{\mathbf{X}} \cdot \left(\mathbf{B}^t \operatorname{cof}(\mathbf{F}) \right) = 0$$

which shows that this vector satisfies an involutive relation in the $\mathbf{X}$ frame. Actually this equation is the simplest one.

Proposition 2. *One has the relation*

$$D_t \left(\operatorname{cof}(\mathbf{F})^{\mathrm{t}} \mathbf{B} \right) = 0. \tag{14}$$

From the identity $\mathbf{F}^t \operatorname{cof}(\mathbf{F}) = \det(\mathbf{F})\,\mathbf{I}$, one obtains

$$D_t \operatorname{cof}(\mathbf{F}) = \mathrm{D_t}\left(\det(\mathbf{F})\right) \mathbf{F}^{-\mathrm{t}} - \mathbf{F}^{-\mathrm{t}} \mathrm{D_t} \mathbf{F}^{\mathrm{t}} \operatorname{cof}(\mathbf{F}).$$

Since $\nabla_{\mathbf{F}} \det(\mathbf{F}) = \operatorname{cof}(\mathbf{F})$ then we obtain the formula

$$D_t \operatorname{cof}(\mathbf{F}) = \left(\operatorname{cof}(\mathbf{F}) : \mathrm{D_t}\mathbf{F} \right) \mathbf{F}^{-\mathrm{t}} - \mathbf{F}^{-\mathrm{t}} \mathrm{D_t} \mathbf{F}^{\mathrm{t}} \operatorname{cof}(\mathbf{F}).$$

At this point we use (6) and obtain the derivative of the product $\mathbf{B}^t \operatorname{cof}(\mathbf{F})$

$$\begin{aligned} D_t \left(\mathbf{B}^t \operatorname{cof}(\mathbf{F}) \right) &= D_t \mathbf{B}^t \operatorname{cof}(\mathbf{F}) + \mathbf{B}^{\mathrm{t}} \mathrm{D_t} \operatorname{cof}(\mathbf{F}) \\ &= \left((\nabla_{\mathbf{x}} \mathbf{u})\, \mathbf{B} - (\nabla_{\mathbf{x}} \cdot \mathbf{u})\, \mathbf{B} \right)^t \operatorname{cof}(\mathbf{F}) \\ &+ \mathbf{B}^t \left(\left(\operatorname{cof}(\mathbf{F}) : \mathrm{D_t}\mathbf{F} \right) \mathbf{F}^{-t} - \mathbf{F}^{-t} D_t \mathbf{F}^t \operatorname{cof}(\mathbf{F}) \right) \\ &= \mathbf{B}^t \left(\nabla_{\mathbf{x}} \mathbf{u}^t - \mathbf{F}^{-t} \nabla_{\mathbf{X}} \mathbf{u}^t \right) \operatorname{cof}(\mathbf{F}) \qquad (= \mathrm{A}_1) \\ &+ \mathbf{B}^t \mathbf{F}^{-t} \left(\operatorname{cof}(\mathbf{F}) : \nabla_{\mathbf{X}} \mathbf{u} - \mathbf{F}^t \operatorname{cof}(\mathbf{F})\, \nabla_{\mathbf{x}} \cdot \mathbf{u} \right) \qquad (= A_2). \end{aligned}$$

The first term A_1 on the right hand side is a matrix applied to $\mathbf{B}^t$. This matrix is actually zero since by application of the chain rule

$$\nabla_{\mathbf{x}} \mathbf{u} = \nabla_{\mathbf{X}} \mathbf{u} \nabla_{\mathbf{x}} \mathbf{X} = \nabla_{\mathbf{X}} \mathbf{u} \mathbf{F}^{-1} \Longrightarrow \nabla_{\mathbf{x}} \mathbf{u}^t - \mathbf{F}^{-t} \nabla_{\mathbf{X}} \mathbf{u}^t = 0$$

So $A_1 = 0$.

The second term A_2 contains the contribution $\mathbf{F}^t \operatorname{cof}(\mathbf{F})\, \nabla_{\mathbf{x}} \cdot \mathbf{u} = \det(\mathbf{F})\, \nabla_{\mathbf{x}} \cdot \mathbf{u}$. The other part is

$$\operatorname{cof}(\mathbf{F}) : \nabla_{\mathbf{X}} \mathbf{u} = \det(\mathbf{F})\, \mathbf{F}^{-\mathrm{t}} : \nabla_{\mathbf{X}} \mathbf{u} = \det(\mathbf{F})\, \nabla_{\mathbf{x}} \mathbf{X}^{\mathrm{t}} : \nabla_{\mathbf{X}} \mathbf{u} = \det(\mathbf{F})\, \nabla_{\mathbf{x}} \cdot \mathbf{u}.$$

Since they subtract to each other then $A_2 = 0$. It finishes the proof of the claim. □

Remark 3. *A simple interpretation of the conservation law (14) is possible. Let us consider the integration of (14) on a generic surface $S_{\mathbf{X}}$, not necessarily closed. Then*

$$0 = D_t \int_{S_{\mathbf{X}}} \operatorname{cof}(\mathbf{F})^{\mathrm{t}} \mathbf{B} \cdot \mathbf{n}_{\mathbf{X}} \mathrm{d}\sigma_{\mathbf{X}} = \mathrm{D}_{\mathrm{t}} \int_{S_{\mathbf{X}}} \mathbf{B} \cdot \operatorname{cof}(\mathbf{F}) \mathbf{n}_{\mathbf{X}} \mathrm{d}\sigma_{\mathbf{X}}.$$

Let us integrate this relation between $t = 0$ and $t = T > 0$. At $t = 0$ $\operatorname{cof}(\mathbf{F}) = \mathbf{I}$. One has the well known equality $\operatorname{cof}(\mathbf{F}) \mathbf{n}_{\mathbf{X}} \mathrm{d}\sigma_{\mathbf{X}} = \mathbf{n}_{\mathbf{x}} \mathrm{d}\sigma_{\mathbf{x}}$, see [7]. So we find out the equality

$$\int_{S_{\mathbf{x}}(T)} \mathbf{B} \cdot \mathbf{n} d\sigma = \int_{S_{\mathbf{x}}(0)} \mathbf{B} \cdot \mathbf{n} d\sigma \tag{15}$$

where we have used simplified notations. That is: the magnetic flux is constant through a lagrangian surface. This is the classical frozen law [16]. Notice also that writing (15) for all lagrangian surface $S_{\mathbf{x}}(0)$ gives back (14). Therefore (14) and (15) are two different formulations of the frozen law.

It is convenient to introduce a new vector

$$\mathbf{C} = \operatorname{cof}(\mathbf{F})^{\mathrm{t}} \mathbf{B} \tag{16}$$

such that the equation (14) rewrites as $D_t \mathbf{C} = 0$. Since $\mathbf{C}$ is now constant in time, then the magnetic field is a simple function of $\mathbf{C}$ and of the deformation gradient $\mathbf{F}$

$$\mathbf{B} = \operatorname{cof}(\mathbf{F})^{-\mathrm{t}} \mathbf{C} = \frac{\mathbf{F}\mathbf{C}}{\det(\mathbf{F})}. \tag{17}$$

In view of the hyperelastic formulation (8) it is natural to introduce potential

$$\varphi^M(\mathbf{F}, S, \rho_0, \mathbf{C}) = \varepsilon_g\left(\frac{\det(\mathbf{F})}{\rho_0}, S\right) + \frac{|\mathbf{F}\mathbf{C}|^2}{2\rho_0\mu_0 \det(\mathbf{F})}, \tag{18}$$

which is also equal to the total energy minus the kinetic energy. The superscript ^{M} is here to indicate the magnetic nature of the potential. The specific volume is $\tau = \frac{1}{\rho} = \rho_0 \det(\mathbf{F})$. With this notation $\varepsilon_g\left(\frac{\det(\mathbf{F})}{\rho_0}, S\right) = \varepsilon_g(\tau, S)$ which means that this part of the potential is the classical internal energy for a gas. One can assume a perfect gas internal energy law

$$\varepsilon_g(\tau, S) = \frac{e^S}{\tau^{\gamma-1}}.$$

It is also natural to define the magnetic Piola-Kirkhhoff stress tensor with the same formula as in (10), that is

$$\sigma^M = \rho_0 \nabla_{\mathbf{F}} \varphi^M. \tag{19}$$

Proposition 4. *One has the expression*

$$\sigma^M = -\left(p\mathbf{I} + \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0}\right) \operatorname{cof}(\mathbf{F}). \tag{20}$$

Let us compute the differential $d\varphi^M$ with respect to $\mathbf{F}$. To perform this calculation we rely on two formulas. The first formula is $d(\det(\mathbf{F})) = \mathrm{cof}(\mathbf{F}) : \mathrm{d}\mathbf{F}$. The second one is $d\frac{|\mathbf{FC}|^2}{2} = (\mathbf{FC}, d\mathbf{FC}) = (\mathbf{FC} \otimes \mathbf{C}) : d\mathbf{F}$. Therefore

$$d\varphi^M = \left(\frac{1}{\rho_0} \partial_\tau \varepsilon_g - \frac{|\mathbf{FC}|^2}{2\rho_0 \mu_0 \det(\mathbf{F})^2} \right) \mathrm{cof}(\mathbf{F}) : \mathrm{d}\mathbf{F} + \frac{1}{\rho_0 \mu_0 \det(\mathbf{F})} (\mathbf{FC} \otimes \mathbf{C}) : \mathrm{d}\mathbf{F}.$$

The magnetic tensor (19) is also

$$\begin{aligned} \sigma^M &= \left(\partial_\tau \varepsilon_g - \frac{|\mathbf{FC}|^2}{2\mu_0 \det(\mathbf{F})^2} \right) \mathrm{cof}(\mathbf{F}) + \frac{1}{\mu_0 \det(\mathbf{F})} (\mathbf{FC} \otimes \mathbf{C}) \\ &= \left(\left(\partial_\tau \varepsilon_g - \frac{|\mathbf{FC}|^2}{2\mu_0 \det(\mathbf{F})^2} \right) \mathbf{I} + \frac{1}{\mu_0 \det(\mathbf{F})^2} (\mathbf{FC} \otimes \mathbf{C}) \mathbf{F} \right) \mathrm{cof}(\mathbf{F}) \\ &= \left(\left(\partial_\tau \varepsilon_g - \frac{|\mathbf{FC}|^2}{2\mu_0 \det(\mathbf{F})^2} \right) \mathbf{I} + \frac{1}{\mu_0 \det(\mathbf{F})^2} (\mathbf{FC} \otimes \mathbf{FC}) \right) \mathrm{cof}(\mathbf{F}). \end{aligned}$$

Then we use the second principle of thermodynamics $TdS = d\varepsilon_g + pd\tau$ to get that $\partial_\tau \varepsilon_g = -p$. And finally we replace $\mathbf{FC}$ by $\det(\mathbf{F})\,\mathbf{B}$ everywhere to finish the proof.

Therefore one has the lemma.

Lemma 5. *The lagrangian form of ideal MHD can be recast as a potential system of equation*

$$\begin{cases} D_t \rho_0 = 0, \\ D_t \mathbf{C} = 0, \\ D_t \mathbf{F} = \nabla_{\mathbf{X}} \mathbf{u}, \\ D_t (\rho_0 \mathbf{u}) = \nabla_{\mathbf{X}} \cdot \left(\rho_0 \nabla_{\mathbf{F}} \varphi^M \right), \\ D_t (\rho_0 e) = \nabla_{\mathbf{X}} \cdot \left(\rho_0 \mathbf{u}^t \nabla_{\mathbf{F}} \varphi^M \right). \end{cases} \tag{21}$$

Up to the new variable $\mathbf{C}$ and the definition of the magnetic potential φ^M, the structure of this system is very close to the structure of the hyperelastic model.

3 Hyperbolicity of lagrangian ideal MHD

The hyperbolic properties of the model (21) is related to symetrization of this system and to the entropy property.

Theorem 6. *Assume $\partial_S \varphi_g \neq 0$ and $\rho_0 > 0$. One has the entropy property*

$$D_t S = 0 \tag{22}$$

for smooth solutions to (21), without any free divergence constraint.

It is a well known property of hyperelastic models: for the completeness of this work we redo the proof. Since $\varphi(\mathbf{F}, S, \rho_0, \mathbf{C}) = e - \frac{1}{2}|\mathbf{u}|^2$ then one has

$$\rho_0 \nabla_{\mathbf{F}} \varphi^M : D_t \mathbf{F} + \rho_0 \partial_S \varphi_g D_t S = D_t(\rho_0 e) - \mathbf{u} \cdot D_t(\rho_0 \mathbf{u})$$

so

$$\rho_0 \partial_S \varphi_g D_t S = \nabla_{\mathbf{X}} \cdot \left(\rho_0 \mathbf{u}^t \nabla_{\mathbf{F}} \varphi^M\right) - \mathbf{u} \cdot \nabla_{\mathbf{X}} \cdot \left(\rho_0 \nabla_{\mathbf{F}} \varphi^M\right) - \rho_0 \nabla_{\mathbf{F}} \varphi^M : \nabla_{\mathbf{X}} \mathbf{u} = 0.$$

With the hypotheses $\rho_0 \partial_S \varphi_g \neq 0$ the proof is finished. $\square$

What is surprising is that the entropy property (22) can be checked without using the free divergence property of the magnetic property. It is the combined use of the deformation gradient and the new variable $\mathbf{C}$ which is the reason. Of course if one desires to have a correct definition of the Euler-Lagrange correspondence then the involutive equations for the deformation gradient must be used. But as far as one wants to write everything in the Lagrange variable the involutive constraint on the magnetic field can be forgotten. It makes a fundamental difference with the eulerian formulations (1) and (4).

The entropy property implies the hyperbolicity provided the entropy functional is strictly convex. For hyperelasticity it is well known that the entropy is only polyconvex. For (21) the entropy cannot be strictly convex with respect to $\mathbf{F}$. It is sufficient to have a look to (18) to understand that the convexity of φ^M with respect to $\mathbf{F}$ is restricted to 4 independent quantities which are $\det(\mathbf{F})$ and the 3 components of $\mathbf{FC}$. Instead we study the acoustic magnetic tensor $\mathbf{A}^M(\mathbf{n}) = \left(a_{ij}^M(\mathbf{n})\right)_{1\le i,j\le 3} = \left(\mathbf{A}^M(\mathbf{n})\right)^t$ in the direction $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ which is an immediate generalization of the acoustic tensor for hyperelastic models

$$a_{ij}^M(\mathbf{n}) = \sum_{1\le k,l\le 3} n_k n_l \frac{\partial^2 \varphi^M}{\partial F_{il} \partial F_{jk}} = a_{ji}^M(\mathbf{n}). \tag{23}$$

Theorem 7. *Assume $\partial_S \varphi_g \neq 0$ and $\rho_0 > 0$. The system (21) is hyperbolic in direction $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ if and only if the acoustic magnetic tensor is such that*

$$\mathbf{A}^M(\mathbf{n}) > 0. \tag{24}$$

A detailed proof in the context of hyperelastic models may be found in [14]. See also [7] but without proof. In our case the proof proceeds as follows. The hyperbolicity can be studied using any quasilinear reformulation of the system (24). Since the entropy property (22) holds true unconditionally (that is without the use of a free divergence property) then

$$\begin{cases} D_t \rho_0 = 0, \\ D_t \mathbf{C} = 0, \\ D_t \mathbf{F} = \nabla_{\mathbf{X}} \mathbf{u}, \\ \rho_0 D_t \mathbf{u} = \nabla_{\mathbf{X}} \cdot \left(\rho_0 \nabla_{\mathbf{F}} \varphi^M\right), \\ D_t S = 0, \end{cases}$$

is a quasilinear reformulation of (22). So (22) is hyperbolic in direction $\mathbf{n}$ if and only if the system

$$\begin{cases} D_t \rho_0 = 0, \\ D_t \mathbf{C} = 0, \\ D_r S = 0, \\ D_t F_{ij} = \partial_{\mathbf{X}_j} u_i, \\ D_t u_i = \sum_j \frac{\partial^2 \varphi^M}{\partial F_{ij} \partial F_{kl}} \partial_{\mathbf{X}_j} F_{kl} + \frac{\nabla_{\mathbf{F}} \varphi^M}{\rho_0} \nabla \rho_0 + \nabla_{\mathbf{F}} \partial_S \varphi^M \nabla S, \end{cases}$$

is hyperbolic in direction $\mathbf{n}$. We define two matrices

$$M_1 = (\delta_{ik}\delta_{jl} n_l)_{1 \le i,j \le 3,\ 1 \le k \le 3} \in \mathbb{R}^{9\times 3}$$

and

$$M_2 = \left(\sum_k \sum_l \frac{\partial^2 \varphi^M}{\partial F_{ij} \partial F_{kl}} n_j \right)_{1 \le i \le 3,\ 1 \le k,l \le 3} \in \mathbb{R}^{3\times 9}.$$

Then eigenvalue problem that we need to study writes

$$Q \begin{pmatrix} \overline{\rho_0} \\ \overline{S} \\ \overline{\mathbf{F}} \\ \overline{\mathbf{u}} \end{pmatrix} = \lambda \begin{pmatrix} \overline{\rho_0} \\ \overline{S} \\ \overline{\mathbf{F}} \\ \overline{\mathbf{u}} \end{pmatrix}, \quad Q = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & M_1 \\ \alpha & \beta & M_2 & 0 \end{array} \right) \in \mathbb{R}^{14}, \quad \alpha, \beta \in \mathbb{R}^3. \tag{25}$$

The exact value of $\alpha, \beta \in \mathbb{R}$ have no influence. The system (22) is hyperbolic if and only if this matrix admits a complete set of real eigenvectors and real eigenvalues. Non zero eigenvalues $\lambda \neq 0$ are such that $M_2 M_1 \overline{\mathbf{u}} = \lambda^2 \rho_0^2 \overline{\mathbf{u}}$ that is

$$\sum_j \left(\sum_{1 \le k,l \le 3} n_k n_l \frac{\partial^2 \varphi^M}{\partial F_{il} \partial F_{jk}} \right) u_j = \lambda^2 u_i \Longleftrightarrow A^M(\mathbf{n}) \overline{\mathbf{u}} = \lambda^2 \overline{\mathbf{u}}. \tag{26}$$

• Assume that all eigenvalues $\mu_i > 0$ for $1 \le i \le 3$ of the acoustic tensor A^M are positive. Then $\lambda = \pm\sqrt{\mu_i}$ corresponds to 6 a non zero eigenvalue of (25). Since the rank of the matrix $(\alpha, \beta, M_2) \in \mathbb{R}^{3\times 11}$ is less or equal to 3, the kernel of M_2 is dimension greater or equal to 8, associated to 8 (or more) real eigenvectors of (25) for the null eigenvalue. So Q admits a set of at least $\ge 6 + 8 = 14$ real eigenvalues and real eigenvectors. In this case (25) is hyperbolic and so is (21).

• On the other hand assume that the acoustic tensor has a null eigenvalue $\mu = 0$ with eigenvector $\overline{\mathbf{u}} \neq 0$, that is $M_2 M_1 \overline{\mathbf{u}} = 0$. Let us examine the vector $Z = (0, 0, M_1 \overline{\mathbf{u}}, \overline{\mathbf{u}})^t \in \mathbb{R}^{14}$. Then by construction

$$QZ = (0, 0, M_1 \overline{\mathbf{u}}, 0)^t \neq 0 \qquad \text{and } Q^2 Z = 0.$$

So the vector Z is in the spectral subspace associated to the eigenvalue 0, but is not an eigenvector. The matrix Q has an Jordan bloc. In this case the problem is not hyperbolic, but only weakly hyperbolic.

Therefore we need to compute the acoustic magnetic tensor to determine the hyperbolicity of the problem (22).

Proposition 8. *One has the formula*

$$\mathbf{A}^M(\mathbf{n}) = \frac{\rho^2}{\rho_0^2}\left(c^2 + \frac{|\mathbf{B}|^2}{\mu_0\rho}\right)(\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}) \otimes (\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}) + \frac{\rho}{\rho_0^2\mu_0}(\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}, \mathbf{B})^2\,\mathbf{I}$$
$$-\frac{\rho\,(\mathrm{cof}\,(\mathbf{F}, \mathbf{B})\,\mathbf{n})}{\rho_0^2\mu_0}(\mathbf{B} \otimes (\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}) + (\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}) \otimes \mathbf{B})\,. \tag{27}$$

One has to characterize (23) with the potential (18). The calculations are a little lengthy but evident. First $\sum_{k,l} n_k n_l \frac{\partial^2 \det(\mathbf{F})}{\partial F_{il}\partial F_{jk}}$ is equal to the determinant of the matrix where the line number i and the line number j have been replaced by the components of the normal. Therefore this term vanishes identically for all i and j. So one gets the classical formula

$$\sum_{k,l} n_k n_l \frac{\partial^2 \varepsilon_g\left(\tau_0 \det\left(\mathbf{F}\right), \mathrm{S}\right)}{\partial F_{il}\partial F_{jk}} = \tau_0^2 \partial_{\tau\tau}\varepsilon_g \sum_{k,l} n_k n_l \mathrm{cof}\,(\mathbf{F})_{\mathrm{il}}\,\mathrm{cof}\,(\mathbf{F})_{\mathrm{jk}}$$

which shows that the contribution of the internal energy of the gas ε_g in (27) is $\frac{\rho^2}{\rho_0^2}c^2\,(\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}) \otimes (\mathrm{cof}\,(\mathbf{F})\,\mathbf{n})$. We have used the standard definition of the gas sound velocity $\partial_{\tau\tau}\varepsilon_g = \rho^2 c^2 > 0$. It remains to study the contribution of the magnetic part $\frac{|\mathbf{FC}|^2}{2\rho_0\mu_0 \det(\mathbf{F})}$. The second derivative of $\det(\mathbf{F})$ is the reason of the magnetic contribution $\frac{|\mathbf{B}|^2}{\mu_0\rho}$ in (27). The second derivative of $\frac{|\mathbf{FC}|^2}{2}$ with respect to $\mathbf{F}$ is also evident to compute. And finally the mixed derivative, that is the first gradient of $\frac{|\mathbf{FC}|^2}{2}$ tensorized by the first gradient of $\det(\mathbf{F})$ (and after that symetrized), is equal to the last line in (27). $\square$

Notice that $\mathbf{A}^M(\mathbf{n}) = \frac{\rho^2}{\rho_0^2 P^2}\mathbf{A}_E^M(\mathbf{n_x})$ where $\mathbf{A}_E^M(\mathbf{n_x})$ is the eulerian matrix

$$\mathbf{A}_E^M(\mathbf{n_x}) = \left(c^2 + \frac{|\mathbf{B}|^2}{\mu_0\rho}\right)\mathbf{n_x} \otimes \mathbf{n_x} + \frac{(\mathbf{n_x}, \mathbf{B})^2}{\rho\mu_0}\mathbf{I} - \frac{(\mathbf{n_x}, \mathbf{B})}{\rho\mu_0}(\mathbf{B} \otimes \mathbf{n_x} + \mathbf{n_x} \otimes \mathbf{B}) \tag{28}$$

and $(\mathbf{n_x})$ is the eulerian normal such that $\mathbf{n_x} = P\,\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}$, $P \in \mathbb{R}$, which means that the eulerian normal $\mathbf{n_x}$ is parallel to the lagrangian one premultipled by the comatrix.

Let us consider for convenience an orthonormal basis $(\mathbf{n}, \mathbf{t}_1, \mathbf{t_2})$ such that

$$\mathbf{B} = \alpha\mathbf{n_x} + \beta\mathbf{t}_1, \qquad \alpha = (\mathbf{n_x}, \mathbf{B})\,, \quad |\mathbf{B}|^2 = \alpha^2 + \beta^2.$$

Proposition 9. *The eigenvalues* $0 \le \lambda_s^2 \le \lambda_a^2 \le \lambda_f^2$ *of the acoustic tensor* $\mathbf{A}_E^M$ *are*

$$\lambda_s^2 = \frac{1}{2}\left(a^2 - \sqrt{a^4 - 4\frac{c^2\alpha^2}{\rho\mu_0}}\right), \quad \lambda_a^2 = \frac{\alpha^2}{\rho\mu_0}, \quad \lambda_f^2 = \frac{1}{2}\left(a^2 - \sqrt{a^4 - 4\frac{c^2\alpha^2}{\rho\mu_0}}\right),$$

where $a^2 = c^2 + \frac{|\mathbf{B}|^2}{\mu_0\rho}$.

In the chosen basis $\mathbf{A}_E^M(\mathbf{n}_\mathbf{x})$ rewrites as

$$\mathbf{A}_E^M(\mathbf{n}_\mathbf{x}) = \begin{pmatrix} c^2 + \frac{\beta^2}{\mu_0 \rho} & -\frac{\alpha\beta}{\rho\mu_0} & 0 \\ -\frac{\alpha\beta}{\rho\mu_0} & \frac{\alpha^2}{\rho\mu_0} & 0 \\ 0 & 0 & \frac{\alpha^2}{\rho\mu_0} \end{pmatrix}.$$

The rest of the computations are evident. □

The eigenvalues λ_s, λ_a and λ_f are the slow, Alfven and fast wave velocities of ideal MHD. At this point of the study, the following theorem is evident.

Theorem 10. *The system lagrangian MHD system (21) is hyperbolic in lagrangian all directions* $\mathbf{n}$ *such that* $(\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}, \mathbf{B}) \neq 0$*. If* $(\mathrm{cof}\,(\mathbf{F})\,\mathbf{n}, \mathbf{B}) = 0$ *the system is only weakly hyperbolic, that is the eigenvalues are all real but some eigenvectors miss.*

With the eulerian normal the condition for strong hyperbolicity writes

$$(\mathbf{n}_\mathbf{x}, \mathbf{B}) \neq 0.$$

The condition for hyperbolicity in a given direction reduces to $\lambda_f^2 > 0$, that is $\alpha \neq 0$. The proof is ended.

An interpretation of this condition is possible based on the following considerations. Strong hyperbolicity implies that the Cauchy problem is well posed for a finite time [7]. Weak formulation implies that the regularity of the solution of the Cauchy suffers for a loss of derivatives.

If one transposes this to the solution of the Riemann problem for the lagrangian system (21) , it means that the solution of a strongly hyperbolic formulation admits bounded solutions, while the solution of a strongly hyperbolic formulation admits measured valued solutions.

This is compatible with the standard classification of discontinuous solutions to ideal MHD [16]. Indeed if $(\mathbf{n}_\mathbf{x}, \mathbf{B}) = 0$ then the tangential components of the velocity are discontinuous [16]. In this case the gradient of deformation is measure valued. But if $(\mathbf{n}_\mathbf{x}, \mathbf{B}) \neq 0$ then the tangential velocities are continuous, which does not generate a measure valued deformation gradient. In summary the condition $(\mathbf{n}_\mathbf{x}, \mathbf{B}) = 0.$ is linked to shear flows.

Remark 11. *It must be emphasized that it is specific to lagrangian systems. Eulerian ideal MHD is hyperbolic (even if the notion of hyperbolicity must take into account the free divergence condition on* $\mathbf{B}$ *to be relevant).*

Remark 12. *It is also possible to include in the potential an hyperelastic part, for example the sum of (9) and of the magnetic part of (18)*

$$\varphi^F = \psi(i_1, i_2, i_3, S, \rho_0) + \frac{|\mathbf{FC}|^2}{2\rho_0\mu_0 \det(\mathbf{F})}.$$

We readily obtain a model for the interaction of a strong magnetic field and material strength that we do not discuss in detail. The final model is hyperbolic once the hyperelastic part is. This is often the case for standard materials modeled by a polyconvex equation of state [7, 14]. Physically this is because standard elastic material cannot have pure shear flows.

4 Eulerian formulations of ideal magnetohydrodynamics

Finally we write an alternative eulerian formulation of (21)

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{C}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{C} \otimes \mathbf{u}) = 0 \\ \partial_t \mathbf{F}^{-1} + \nabla_{\mathbf{x}} \left(\mathbf{F}^{-1} \mathbf{u} \right) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \nabla_{\mathbf{x}} \cdot \sigma, \\ \partial_t (\rho e) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} e) = \nabla_{\mathbf{x}} \cdot (\mathbf{u}^t \sigma) . \end{cases} \tag{29}$$

This is completely standard in the context of hyperelastic models [17]. Here the stress tensor is

$$\sigma = \sigma^M \mathbf{F}^t = -p\mathbf{I} - \frac{|\mathbb{B}|^2}{2\mu_0}\mathbf{I} + \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0}.$$

The equation for the magnetic field $\partial_t B - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = 0$ has been replaced by a series of transport-like equation

$$\begin{cases} \partial_t (\rho \mathbf{C}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{C} \otimes \mathbf{u}) = 0 \\ \partial_t \mathbf{F}^{-1} + \nabla_{\mathbf{x}} \left(\mathbf{F}^{-1} \mathbf{u} \right) = 0. \end{cases}$$

It is also possible [7, 14] to use the equation

$$\partial_t \left(\frac{\mathbf{F}}{\det \mathbf{F}} \right) + \nabla \cdot \left(\frac{\mathbf{F} \otimes \mathbf{u}}{\det \mathbf{F}} \right) = \nabla \cdot \left(\frac{\mathbf{u} \otimes \mathbf{F}^t}{\det \mathbf{F}} \right) \tag{30}$$

instead of $\partial_t \mathbf{F}^{-1} + \nabla_{\mathbf{x}} \left(\mathbf{F}^{-1} \mathbf{u} \right) = 0$.

5 Numerical perspectives

All these non standard formulations of lagrangian and eulerian ideal MHD may be of interest for the design of numerical methods.

More specifically we have in mind to solve (1) with a Lagrange+remap approach, where the time step is decomposed in two stages. In the first stage one uses a Lagrange formulation of the equation written in the comobile frame as in [2]. See also [14] for the discretization of non linear hyperelastic models. For this stage one can use (21). Then in the second stage one remaps the moving mesh onto the old one. This second stage is very easy. For example an upwind first-order discretization with the donor-cell method is stable. So the whole problem is the stability of the algorithm in the Lagrange phase. It is therefore reasonable to think that theorem 6 may have immediate application to the design of entropy compatible lagrangian discrete schemes without any need of a discrete free divergence condition of the magnetic field. The eulerian formulation (29-30) can also be used for numerical simulations of ideal MHD as an alternative to (1) or (4), using for example the recent eulerian numerical methods for hyperelasticity developed in [9, 1]. We leave these issues for further studies.

References

[1] D.J. Barton, D. Drikakis, and E.I. Romenski. An Eulerian finite-volume scheme for large elastoplastic deformations in solids. *Int.J. for Num.Meth. in Eng., online*, 2009.

[2] F. Bezard and B. Després. *Journ. Comp. Phys.*, 154-1:65–89, 1999.

[3] F. Bouchut, K. Klingenberg, and K. Waaga. Ann approximate Riemann solver for ideal MHD based on relaxation. In *Hyperbolic problems. Theory, numerics ans applications. In proceedings of the 12th international conference on hyperbolic problems*, University Maryland, College Park, 2008.

[4] J. U. Brackbill and D. C. Barnes. The effect of no zero $\nabla \cdot \mathbf{b}$ on the numerical solution of the magnetohydrodynamics equations. *J. Comput. Phys.*, 35:426–430, 1980.

[5] Y. Brenier. Hydrodynamic structure of the augmented Born-Infeld equations. *ARMA*, 172:65–91, 2004.

[6] P. Cargo and P. Gallice. Roe matrices for ideal mhd and systematic construction of roe matrices for systems of conservation laws. *J. of Comp. Physics*, 136:446–466, 1997.

[7] C. Dafermos. *Hyperbolic conservation laws in continuum physics*. Springer, 2005.

[8] W Dai and P. R. Woodward. On the divergence-free condition and conservaton laws in numerical simulations for supersonic magnetohydrodynamic flows. *Astrophys. J.*, 494:317–335, 1998.

[9] N. Favrie, S. L. Gavrilyuk, and R. Saurel. Solid fluid diffuse interface model in cases of extreme deformations. *J. Comput. Phys.*, 228:6037–6077, 2009.

[10] Gallice G. Positive and entropy stable godunov-type schemes for gas dynamics and mhd equations in lagrangian or eulerian coordinates. *Numer. Math.*, (94), 2003.

[11] S. K. Godunov and E. I. Romensky. Thermodynamics, conservation laws and symmetric forms of differential equations in mechanics of continuous media. *in Comput. Fluid Dynamics Review*, pages 19–31, 1995.

[12] S.K. Godunov. Lois de conservation et intégrales d'énergie des équations hyperboliques. volume 1270. Springer Verlag, 1986.

[13] M. Kiessling. Electromagnetic field theory without divergence problems. I. The Born legacy. *J. Statist. Phys.*, 116:1057–1122, 2004.

[14] G. Kluth and B. Després. Discretization of hyperelasticity on unstructured mesh with a cell centered lagrangian scheme. Technical Report R09063, LJLL-UPMC, 2009.

[15] A. G. Kulikovskii, Pogorelov N. V., and A. Y. Semenov. *Mathematical aspects of numerical solution of hyperbolic systems*. Chapman-Hall, 2001.

[16] L Landau and E. Lifshitz. *Physique théorique, Electrodynamique des milieux continus*. MIR, 1990.

[17] G. H. Miller and P Colella. A conservative thre-dimensional eulerian method for coupled solidfluid shock capturing. *J. Comp. Phys.*, pages 26–82, 2002.

[18] K. Powell, P. L. Roe, T. J. Linde, T. I. Gombosi, and D. zeeuw. A solution-adpatative upwind scheme for idela magnetohydrodynamics. *J. of Comp. Physics*, 154:284–309, 1999.

[19] K. G. Powell. An approximate Riemann solver for magnetohydrodynamics (that works in more than one dimension). Technical report, ICASE, 1994.

[20] P. L. Roe and D. S. Balsara. Notes on the eigensystem of magnetohydrodynamics. *Siam J. Appl. Math.*, 56:56–67, 1996.

[21] D. Serre. Hyperbolicity of the non linear models of Maxwell's equations. *ARMA*, 172:309–333, 2004.

[22] D. H. Wagner. Symmetric hyperbolic equations of motion for a hyperelastic material. *to appear*.

[23] A Zachary, A. Malagoli, and P. Colella. A high-order godunov method for multidimensional magnetohydrodynamics. *Siam J. Sci. Comp.*, 15:263–284, 1994.